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# ON THE STRUCTURE OF $n$-POINT SETS

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ABSTRACT

Let $n$ be an integer greater than one. Our main result, called the "Structure Theorem" is that a set that contains $n-1$ disjoint continua that are cut by a single line cannot be an $n$-point set, that is, a set that meets every line in precisely $n$ points. This theorem unifies and significantly improves upon a number of known theorems. The second part of the paper is devoted to several theorems that address the question when a set that meets every line in at most $n$ points can be extended to an $n$-point set. These theorems also highlight the sharpness of the Structure Theorem.

## 1. Introduction

If $n$ is an integer greater than one then an **$n$-point set** is a subset of the plane that meets every line in precisely $n$ points. A planar set is called a **partial $n$-point set** if it meets every line in at most $n$ points. The existence of $n$-point sets is implicit in Mazurkiewicz [16] and was formally established in [1] and [19]. A good source of information about these sets is Mauldin [15].

An **arc** is a space that is homeomorphic to the interval $I = [0, 1]$. It has long been known that two-point sets cannot contain arcs; see Larman [14]. In an

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attempt to generalize this fact to $n$-point sets Bouhjar, Dijkstra, and van Mill [4] were surprised to find that four-point sets can contain arcs whereas three-point sets do not contain arcs. The primary aim of this paper is to present an overarching theorem, which we like to call the Structure Theorem for $n$-Point Sets, that gives in a sense a precise count of the number of arcs that an $n$-point set can contain and thereby provides an explanation for the phenomena discovered in [4].

A **side** of a line $\ell$ in the plane is a component of $\mathbb{R}^2 \setminus \ell$. We say that a line $\ell$ **cuts** a subset $A$ of $\mathbb{R}^2$ if $A$ contains points on both sides of $\ell$.

STRUCTURE THEOREM 1: *If $n \geq 2$ and $X$ is an $n$-point set, then every line cuts at most $n-2$ pairwise disjoint subcontinua of $X$.*

If $u$ is a point in a space $X$ then we say that $X$ is **zero-dimensional at** $u$ (notation $\operatorname{ind}_u X = 0$) if $u$ has a neighbourhood basis in $X$ consisting of clopen sets. A nonempty space is zero-dimensional (notation $\operatorname{ind} X = 0$) if it is zero-dimensional at each of its points.

The following result is a somewhat more technical but stronger version of the Structure Theorem.

STRUCTURE THEOREM 2: *Let $n \geq 2$ and $X$ be an $n$-point set that contains $n-2$ pairwise disjoint continua $C_1, \ldots, C_{n-2}$ that are all cut by some line $\ell$. If $u \in \ell \cap X \setminus \bigcup_{i=1}^{n-2} C_i$ then $\operatorname{ind}_u X = 0$.*

§2 of this paper is devoted to the proof of these theorems. In §3 we present a number of extension theorems and examples that highlight the sharpness of the structure theorems.

## 2. The structure theorems

We start with a few definitions. We shall use the standard inner product and norm on the vector space $\mathbb{R}^2$: $(x, y) \cdot (x', y') = xx' + yy'$ and $\|u\| = \sqrt{u \cdot u}$. The origin is denoted by $\mathbf{0}$. If $X \subset \mathbb{R}^2$ then $\operatorname{int} X$, $\partial X$, and $\overline{X}$ denote respectively the interior, the boundary, and the closure of $X$ in the plane. If $\varepsilon > 0$ and $A \subset \mathbb{R}^2$ then $U_\varepsilon(A) = \{u \in \mathbb{R}^2 \colon \|u - v\| < \varepsilon \text{ for some } v \in A\}$. If $u$ and $v$ are distinct points of $\mathbb{R}^2$ then $L(u, v)$ denotes the line through $u$ and $v$. Let $\mathbb{P}$ be the projective line. If a polar coordinate system has been chosen for the plane then we identify $\mathbb{P}$ with the circle group $\mathbb{R}/\pi\mathbb{Z}$. If $\ell$ is a line in the plane then $\theta(\ell) \in \mathbb{P}$ denotes the angle of inclination of $\ell$. The cardinality of a set $X$ is denoted by $|X|$.

A space is called **rim-finite** if there is a basis for the topology consisting of sets with finite boundaries. Obviously, every partial $n$-point set is rim-finite (and hence at most one-dimensional). According to Menger [17, pp. 56, 256] every rim-finite continuum is arcwise connected so a (partial) $n$-point contains a nontrivial continuum if and only if it contains an arc.

This section is devoted to the proof of Theorems 1 and 2. Let us first mention several known theorems that are corollaries of the structure theorems. The following result was first proved by Larman [14] and is equivalent to Theorem 1 with $n = 2$ substituted.

LEMMA 3: *No two-point set contains arcs.*

The next result is due to Kulesza [12] and is precisely the Second Structure Theorem for $n = 2$. This is no surprise because Kulesza's proof is the main ingredient of the argument that leads us from the First Structure Theorem to the Second.

THEOREM 4: *Every two-point set is zero-dimensional.*

The following theorem is the main result in Bouhjar, Dijkstra, and van Mill [4]. It is equivalent to both structure theorems with the substitution $n = 3$.

THEOREM 5: *No three-point set contains arcs.*

*Proof:* With Menger [17] we obviously have that this theorem implies both structure theorems for $n = 3$.

In order to show that Structure Theorem 1 implies Theorem 5 let $X$ be a three-point set and let $\alpha: I \to X$ be an embedding. Put $p = \alpha(0)$ and $q = \alpha(1)$ and let $\ell$ be a line parallel to $L(p,q)$, that intersects the arc $A = \alpha([0,1])$, and with maximum distance towards $L(p,q)$. Let $t \in (0,1)$ be such that $\alpha(t) \in \ell \cap A$ and select $r \in (0,t)$ close enough to $t$ such that $\alpha(r)$ lies on the same side of $L(p,q)$ as $\alpha(t)$. Let $\ell'$ be a line parallel to $L(p,q)$ that separates $L(p,q)$ from $\alpha(r)$ and $\alpha(t)$. Then $\ell'$ cuts the disjoint continua $\alpha([0,r])$ and $\alpha([t,1])$, in violation of Theorem 1. ∎

The following result was obtained by Bouhjar, Dijkstra, and Mauldin [3] using results from Mauldin [15].

THEOREM 6: *No $n$-point set is an $F_\sigma$-set in the plane.*

Let $X \subset \mathbb{R}^2$ be such that $X$ intersects every vertical line in exactly $n$ points. Then there exist unique functions $\xi_1, \ldots, \xi_n \colon \mathbb{R} \to \mathbb{R}$ such that $\xi_1(x) < \cdots < \xi_n(x)$ for every $x \in \mathbb{R}$ and $X$ equals the union of the graphs of the $\xi_i$'s. Proposition 3.2 in [4] states that if $X$ is an $F_\sigma$-set then there exists a non-degenerate interval $[a, b]$ such that every $\xi_i | [a, b]$ is continuous. Theorem 6 was obtained in [3] by showing in essence that if there exists a line in *every* direction that cuts $n$ pairwise disjoint arcs in $X$ then $X$ cannot be an $n$-point set. Both structure theorems are substantial improvements over this result.

We now present three technical lemmas that will be used extensively in the proof of Theorem 1.

If $\ell$ is a line and $X$ is a subset of the plane then we say that a point $u \in \ell \cap X$ is a **crossing** of $\ell$ with $X$ if every neighbourhood of $u$ in $X$ is cut by $\ell$.

LEMMA 7: *If a line $\ell$ cuts a connected set $A$ in the plane and* $\operatorname{ind}(A \cap \ell) = 0$ *(e.g., if $A$ is a partial $n$-point set) then there exists a crossing of $\ell$ with $A$.*

*Proof:* Let $C_1$ and $C_2$ be the two components of $\mathbb{R}^2 \setminus \ell$ and let $F_i$ be the closure in $A$ of $C_i \cap A$ for $i = 1, 2$. Define the open set $U = A \setminus (F_1 \cup F_2)$ in $A$ and note that $U \subset A \cap \ell$. If $a \in U$ then select a closed neighbourhood $F$ of $a$ in $A$ such that $F \subset U$. Since $\operatorname{ind} U \leq \operatorname{ind}(A \cap \ell) = 0$ we can find a clopen subset $O$ of $U$ with $a \in O \subset F$. Then $O$ is clopen in the connected space $A$ and hence $O = A$. So $A \subset U \subset \ell$ which violates the assumption that $\ell$ cuts $A$. Consequently, $U$ must be empty and $F_1 \cup F_2 = A$. Since $A$ is connected we have $F_1 \cap F_2 \neq \emptyset$. Any element of $F_1 \cap F_2$ is a crossing of $\ell$ with $A$. ∎

If $A$ is a subset of the plane and $a \in A$ then we call $a$ an **arc-point** of $A$ if $a$ is contained in an arc that is a subspace of $A$.

LEMMA 8: *Let $n \in \mathbb{N}$, let $O$ be a nonempty open subset of $\mathbb{P}$, and let $u \in A \subset \mathbb{R}^2$. If for every line $\ell$ through $u$ with $\theta(\ell) \in O$, we have* $\operatorname{ind}(\ell \cap A) = 0$ *and $\ell \cap A$ contains at least $n$ arc-points of $A \setminus \{u\}$, then there is a line $\ell$ through $u$ such that $\theta(\ell) \in O$ and $\ell$ cuts at least $n$ pairwise disjoint arcs in $A \setminus \{u\}$.*

*Proof:* Let $\mathcal{L}$ stand for the set of all lines through $a$ with angle of inclination in $O$. Let $m$ be the maximum number of pairwise disjoint arcs in $A \setminus \{a\}$ that are cut by some line in $\mathcal{L}$ and assume that $m < n$. So let $\ell \in \mathcal{L}$ be such that there exist pairwise disjoint arcs $C_1, \ldots, C_m$ in $A \setminus \{a\}$ that are cut by $\ell$. Select with Lemma 7 a crossing $b_i$ of $\ell$ with $C_i$ for each $i \leq m$. Since $m < n$ there is a point $b_0 \in \ell$ and an arc $C_0 \subset A \setminus \{a\}$ such that $b_0 \notin \{b_1, \ldots, b_m\}$ and $b_0 \in C_0$. We may assume without loss of generality that $C_0$ is disjoint from every other

$C_i$. Let the continuous map $p$: $\mathbb{R}^2 \setminus \{a\} \to \mathbb{P}$ be given by $p(u) = \theta(L(a,u))$. Note that for $1 \leq i \leq m$, $\theta(\ell)$ is an interior point of $p(C_i)$ and hence there is an $\varepsilon > 0$ such that the interval $[\theta(\ell) - \varepsilon, \theta(\ell) + \varepsilon]$ is contained in $O \cap \bigcap_{i=1}^m p(C_i)$. Since $\operatorname{ind}(\ell \cap A) = 0$ the line $\ell$ does not contain $C_0$ and hence $p(C_0)$ is a non-degenerate interval that contains $\theta(\ell) = p(b_0)$. Let $\delta > 0$ be such that $\delta < \varepsilon$ and $[\theta(\ell) - \delta, \theta(\ell)]$ or $[\theta(\ell), \theta(\ell) + \delta]$ is contained in $p(C_0)$. See Figure 1.

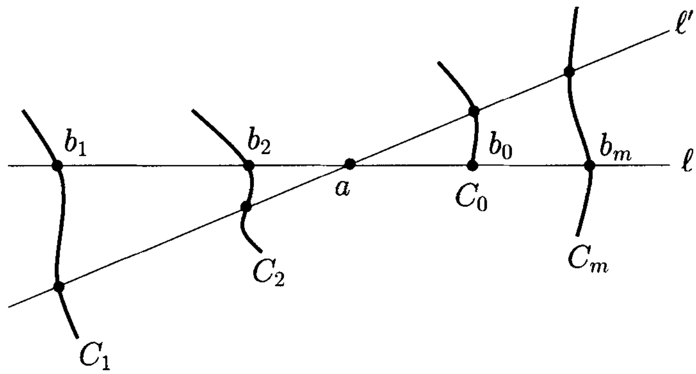


Figure 1

If $\ell'$ is a line through $a$ with $\theta(\ell')$ interior point of whichever interval of the two is contained in $p(C_0)$ then $\ell'$ cuts every $C_i$, $0 \leq i \leq m$, contradicting the maximality of $m$. ∎

Let $\pi_1, \pi_2$: $\mathbb{R}^2 \to \mathbb{R}$ be the projections onto the $x$-axis respectively the $y$-axis. We choose the usual polar coordinate system for the plane, which means that $\pi/2 \in \mathbb{P}$ represents the vertical lines.

*From now on for the remainder of this section let $n$ be a fixed integer greater than one.*

LEMMA 9: *Let $X \subset \mathbb{R}^2$ be such that for each point $u \in X$ there is a neighbourhood $U_u$ of $\pi/2$ in $\mathbb{P}$ such that for each line $\ell$ through $u$ with $\theta(\ell) \in U_u$ we have $|\ell \cap X| \leq n$. If the $y$-axis cuts $n-1$ pairwise disjoint arcs contained in $X$ and if no vertical line intersects $X$ in precisely $n-1$ points then there exist an $\varepsilon > 0$ and $n$ continuous functions $\xi_1 < \ldots < \xi_n$: $(-\varepsilon, \varepsilon) \to \mathbb{R}$ such that $X \cap ((-\varepsilon, \varepsilon) \times \mathbb{R})$ is the union of the graphs of the $\xi_i$'s.*

*Proof:* Note that the assumptions imply that every vertical line intersects $X$ in either exactly $n$ points or in fewer than $n-1$ points. Let $C_1, \ldots, C_{n-1}$ be pairwise disjoint arcs in $X$ that are cut by the $y$-axis. Let $\varepsilon > 0$ be such that

$[-\varepsilon,\varepsilon] \subset \bigcap_{i=1}^{n-1} \pi_1(C_i)$. Let $i \in \{1,\ldots,n-1\}$. We select an embedding $\alpha\colon I \to C_i$ such that $\pi_1 \circ \alpha(0) = -\varepsilon$ and $\pi_1 \circ \alpha(1) = \varepsilon$. Define $s = \min(\pi_1 \circ \alpha)^{-1}(\{\varepsilon\})$ and $r = \max([0,s] \cap (\pi_1 \circ \alpha)^{-1}(\{-\varepsilon\}))$. Then $\pi_1 \circ \alpha([r,s]) = [-\varepsilon,\varepsilon]$, $\pi_1 \circ \alpha(r) = -\varepsilon$, and $\pi_1 \circ \alpha(s) = \varepsilon$. So without loss of generality we may assume that $C_i$ is an arc and that there is an homeomorphism $\alpha_i\colon I \to C_i$ with $\pi_1(C_i) = [-\varepsilon,\varepsilon]$, $\beta_i^{-1}(-\varepsilon) = \{0\}$, and $\beta_i^{-1}(\varepsilon) = \{1\}$, where $\beta_i = \pi_1 \circ \alpha_i$.

Next we show that $\beta_i$ is one-to-one. If $\beta_i\colon I \to [-\varepsilon,\varepsilon]$ were not strictly increasing then by continuity there would exist a $t \in (-\varepsilon,\varepsilon)$ such that the fibre $\beta_i^{-1}(\{t\})$ contains at least three points which means that the vertical line $\ell = \{t\} \times \mathbb{R}$ would intersect $C_i$ in at least three points. Since $\ell$ meets also every other $C_j$ this result would mean that $\ell$ meets $X$ in at least $n+1$ points in contradiction to the assumption that every vertical line meets $X$ in at most $n$ points. So we have that $\pi_1|C_i\colon C_i \to [-\varepsilon,\varepsilon]$ is a homeomorphism and we let $\psi_i\colon [-\varepsilon,\varepsilon] \to \mathbb{R}$ denote the continuous map $\pi_2 \circ (\pi_1|C_i)^{-1}$. Note that $C_i$ is the graph of $\psi_i$.

If $x \in [-\varepsilon,\varepsilon]$ then $X$ intersects the vertical line $\{x\} \times \mathbb{R}$ in precisely $n$ points: $(x,\psi_1(x)),\ldots,(x,\psi_{n-1}(x))$ and one other point which we will denote by $(x,\psi_n(x))$. So $\psi_n\colon [-\varepsilon,\varepsilon] \to \mathbb{R}$ is a well-defined function and the graphs of all the $\psi_i$'s are pairwise disjoint.

We prove that $\psi_n|(-\varepsilon,\varepsilon)$ is continuous. Let $a \in (-\varepsilon,\varepsilon)$ and let $a_1,a_2,\ldots$ be a sequence in $(-\varepsilon,\varepsilon) \setminus \{a\}$ such that $\lim_{j\to\infty} a_j = a$ and $\lim_{j\to\infty} \psi_n(a_j) = b$, where $b \neq \psi_n(a)$ and we allow $b = \pm\infty$. Let $N > 0$ be such that every line through $u = (a,\psi_n(a))$ with slope greater than $N$ or less than $-N$ has its angle of inclination in $U_u$, so it intersects $X$ in at most $n$ points. Define

$$M = N + \max\left\{\left|\frac{\psi_i(s) - \psi_n(a)}{s-a}\right| : i \in \{1,\ldots,n-1\}, s \in \{-\varepsilon,\varepsilon\}\right\}.$$

Choose a $j$ such that $|\psi_n(a_j) - \psi_n(a)| > M|a_j - a|$. If $\ell$ is the line through $(a,\psi_n(a))$ and $(a_j,\psi_n(a_j))$ then its slope is either greater than $M$ or less than $-M$. This means that $\ell$ is steep enough and close enough to the vertical line $\{a\} \times \mathbb{R}$ so that $\ell$ intersects the graph of $\psi_i$ for every $i \leq n-1$; see Figure 2.

Since $\ell$ intersects the graph of $\psi_n$ in two points we find that $\ell \cap X$ contains at least $n+1$ points, which is not possible because the absolute value of the slope of $\ell$ exceeds $N$ and hence $|\ell \cap X| \leq n$. It now follows from the Intermediate Value Theorem that the $\psi_i|(-\varepsilon,\varepsilon)$'s can be ordered to produce the sequence $\xi_1 < \ldots < \xi_n$. ∎

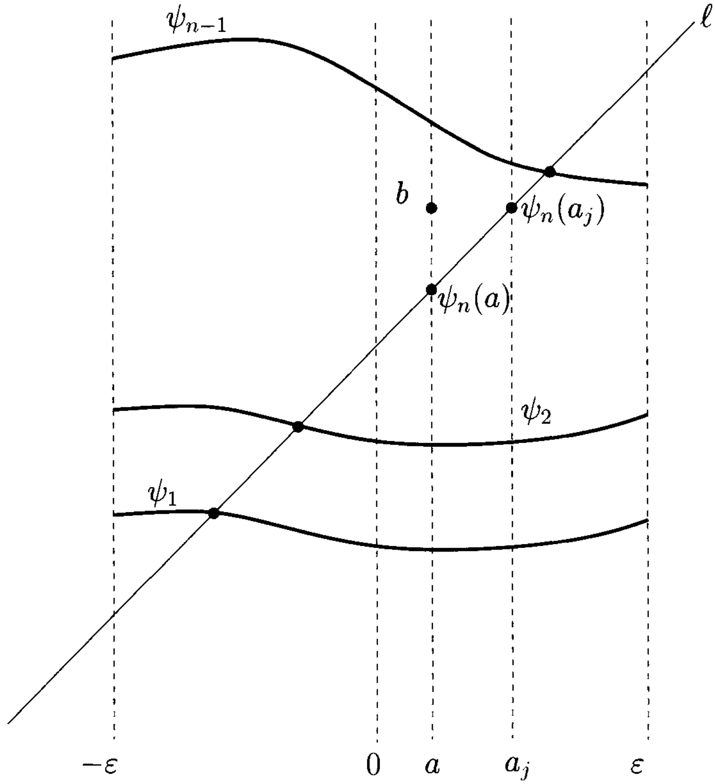


Figure 2

THEOREM 10: *Let $X \subset \mathbb{R}^2$ and let $U$ be a neighbourhood of $\pi/2$ in $\mathbb{P}$ such that every line $\ell$ with $\theta(\ell) \in U$ and $\ell \cap \overline{X} \neq \emptyset$ meets $X$ in precisely $n$ points. If there is a vertical line that cuts $n-1$ disjoint subcontinua of $X$ then there exist $n$ continuous functions $\xi_1 < \cdots < \xi_n \colon \mathbb{R} \to \mathbb{R}$ such that $X$ equals the union of the graphs of the $\xi_i$'s.*

*Proof:* Let $X$ and $U$ be as in the premises of the theorem. Choose an $xy$-coordinate system such the $y$-axis cuts $n-1$ pairwise disjoint subcontinua of $X$.

Since every vertical line either misses $X$ or intersects $X$ in precisely $n$ points there exist uniquely determined functions $\xi_1 < \cdots < \xi_n \colon \pi_1(X) \to \mathbb{R}$ such that $X$ equals the union of the graphs of the functions.

Let $N > 0$ be such that every line $\ell$ that meets $\overline{X}$ and whose slope exceeds $N$ or is less than $-N$ has the property $|\ell \cap X| = n$. Note that every line with slope $> N$ or $< N$ meets $X$ in 0 or $n$ points. It is easily seen that every point in the plane has arbitrarily small neighbourhoods whose boundaries are contained in, say, four of such lines. Consequently, $X$ and its subsets are rim-finite. According to Menger [17] every rim-finite continuum is arcwise connected so we may assume that the $y$-axis cuts $n-1$ pairwise disjoint arcs in $X$. Lemma 9 then guarantees that there is a $\delta > 0$ such that $(-\delta, \delta) \subset \pi_1(X)$ and $\xi_i|(-\delta,\delta)$ is continuous for each $i \in \{1, \ldots, n\}$. We will show that every $\xi_i$ is continuous on the whole real line which means that $X$ is merely a topological sum of $n$ copies of $\mathbb{R}$.

Assume that some $\xi_i$ is undefined or discontinuous somewhere. By symmetry we may assume that the problem occurs for positive $x$. We define $s \in [\delta, \infty)$ by

$$s = \sup\{x \in \mathbb{R}\colon\ \xi_i|[0,x) \text{ is well defined and continuous for every } i\}.$$

Note that each $\xi_i|[0,s)$ is continuous.

CLAIM 1: *$\lim_{x \nearrow s} \xi_i(x) = \gamma_i$ exists for each $i \in \{1, \ldots, n\}$.*

*Proof:* First we prove that $\xi_n|[0,s)$ is bounded above. Assume that $\limsup_{x\nearrow s} \xi_n(x) = \infty$ and select an increasing sequence $x_1, x_2, \ldots$ in $(0,s)$ such that $\lim_{j\to\infty} x_j = s$ and $\lim_{j\to\infty} \xi_n(x_j) = \infty$. Select a $k > 1$ such that

$$\frac{\xi_n(x_k) - \xi_n(x_1)}{x_k - x_1} > \max\left\{N, \frac{\xi_n(x_1) - \xi_1(0)}{x_1}\right\}.$$

Let $f$ be the linear function whose graph is the line $\ell$ through $(x_1, \xi_n(x_1))$ and $(x_k, \xi_n(x_k))$. Note that $f(0) < \xi_1(0)$ and that $f(x_1) = \xi_n(x_1) > \xi_{n-1}(x_1)$. Since $\xi_1 < \cdots < \xi_{n-1}$ we have with the Intermediate Value Theorem that $\ell$ cuts the graph of $\xi_i|[0,x_1]$ for each $i \le n-1$; see Figure 3.

Since $\ell$ meets the graph of $\xi_n$ in at least two points we have that $\ell \cap X$ contains at least $n+1$ points, a contradiction because the slope of $\ell$ exceeds $N$.

So we have that $\xi_n|[0,s)$ is bounded above and by symmetry that $\xi_1|[0,s)$ is bounded below, showing that every $\xi_i|[0,s)$ is bounded. Consider a $\xi_i$ and assume that $a = \liminf_{x\nearrow s} \xi_i(x) < \limsup_{x\nearrow s} \xi_i(x) = b$. This means that $\xi_i$ displays "$\sin(1/x)$ behaviour" at $s$. Let $\ell$ be the line through $u = (s, (a+b)/2)$ with slope $2N$ and observe that $u \in \overline{X}$. Note that $\ell$ separates $(s,a)$ from $(s,b)$ and hence we can select an increasing sequence $x_1, x_2, \ldots$ in $(0,s)$ such that $\lim_{j\to\infty} x_j = s$ and $\ell$ separates $(x_j, \xi_i(x_j))$ from $(x_{j+1}, \xi_i(x_{j+1}))$ for each $j \in \mathbb{N}$. Then the graph of $\xi_i$ intersects $\ell$ in infinitely many points, in violation of the property $|\ell \cap X| = n$. So $a = b$ and the claim is proved. ∎

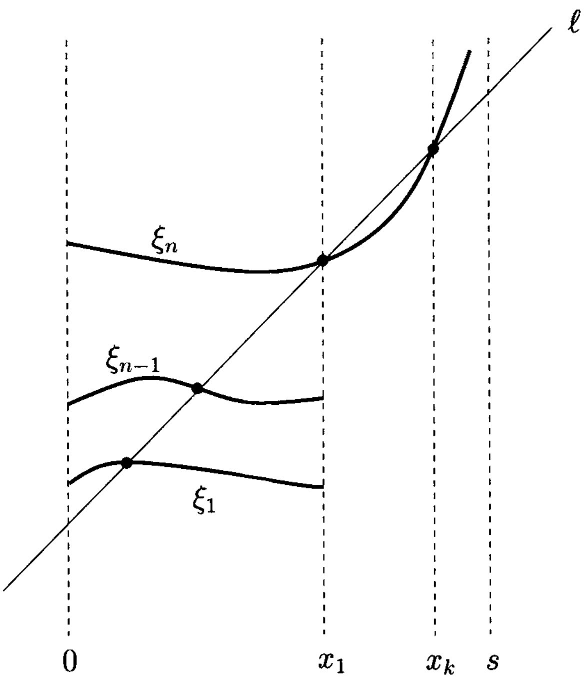


Figure 3

Put $u_i = (s, \gamma_i)$ for $1 \leq i \leq n$ and define

$$m = \max(\{0\} \cup \{i\colon u_i \text{ is no arc-point of } ([s,\infty) \times \mathbb{R}) \cap X\}).$$

Note that $\gamma_1 \leq \cdots \leq \gamma_n$ and choose $\gamma_0 < \gamma_1$ and $\gamma_{n+1} > \gamma_n$. We obviously have $\gamma_m < \gamma_{m+1}$. For every $i \in \{m+1, \ldots, n\}$ let $K_i$ be an arc in $([s,\infty) \times \mathbb{R}) \cap X$ that contains $u_i$. Since $(\{s\} \times \mathbb{R}) \cap X$ is finite, every $\pi_i(K_i)$ is a non-degenerate interval so we may choose an $r > s$ such that $[s, r] \subset \pi_1(K_{m+1}) \cap \cdots \cap \pi_1(K_n)$. Put

$$t = \max(\{0\} \cup \pi_2(K_{m+1} \cup \cdots \cup K_n)).$$

Claim 2: *If $k \in \{m, \ldots, n\}$ is such that $\gamma_k < \cdots < \gamma_n$ then there exist an $M \geq N$ and a dense subset $D$ of $(M, \infty)$ such that for every $\mu \in D$ the line $y = \gamma_k + \mu(x - s)$ cuts $n-1$ pairwise disjoint arcs in $X \setminus \{u_k\}$.*

*Proof:* We may assume that $K_{k+1}, \ldots, K_n$ are pairwise disjoint. We define

$$M = \max\left\{N, \frac{t - \gamma_k}{r - s}, \frac{\gamma_k - \xi_1(0)}{s}\right\}.$$

Let $\mu \in (M, \infty)$ and let the line $\ell_\mu$ be the graph of $f(x) = \gamma_k + \mu(x - s)$. Assume that $a \neq s$ is such that $(a, f(a))$ is a point of $X$ but not an arc-point of $X$. Note that $f(x) = f(a) + \mu(x - a)$, $f(r) > t$, $f(s) = \gamma_k < \gamma_{k+1}$, and $f(0) < \xi_1(0)$. Since $g(\nu, x) = f(a) + \nu(x - a)$ depends continuously on the slope $\nu$ we can find a $\mu' \in (M, \infty)$, arbitrarily close to $\mu$ such that the linear function $h(x) = g(\mu', x)$ shares the following properties with $f(x)$: $h(r) > t$, $h(s) < \gamma_{k+1}$, and $h(0) < \xi_1(0)$. If $a > s$ then we choose $\mu'$ slightly smaller than $\mu$ but greater than $N$ and if $a < s$ then $\mu' > \mu$. See Figure 4 for an illustration of the case $a > s$.

So in either case $h(s) > \gamma_k = \lim_{x \nearrow s} \xi_k(x)$. Let $\ell'$ be the graph of $h$ and note that $\ell'$ cuts the graphs of $\xi_1, \ldots, \xi_k$. Since $h(s) < \gamma_{k+1}$ and $h(r) > t$, $\ell'$ also cuts each of the pairwise disjoint arcs $K_{k+1}, \ldots, K_n$. None of the crossings can be equal to $(a, f(a))$ since that point is no arc-point of $X$. So we have $|\ell' \cap X| \geq n+1$, a contradiction because $\mu' > N$.

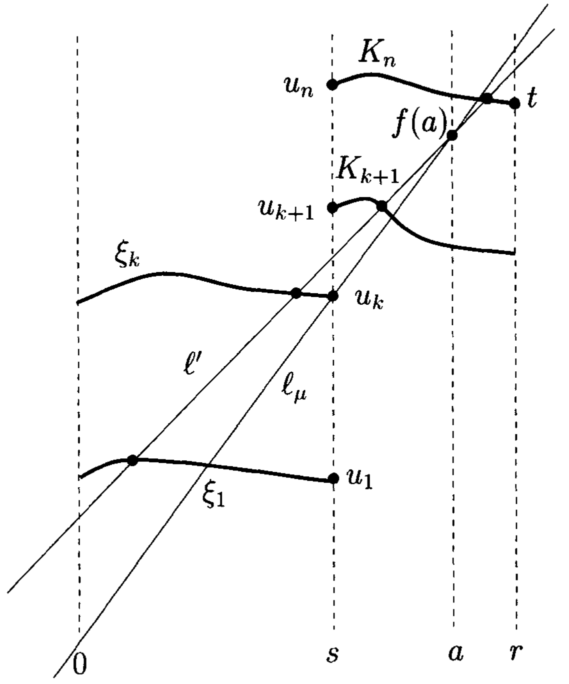


Figure 4

We may conclude that every point in $\ell_\mu \cap X \setminus \{u_k\}$ is an arc-point of $X \setminus \{u_k\}$ for any $\mu \in (M, \infty)$. Since $\mu > N$ and $u_k \in \ell \cap \overline{x}$ we have $|\ell_\mu \cap X| = n$ so $\ell_\mu$ must contain at least $n-1$ points from $X \setminus \{u_k\}$, all of which are arc-points. Applying Lemma 8 we find that there exists a $\nu$, arbitrarily close to $\mu$, such that $\ell_\nu$ cuts $n-1$ pairwise disjoint arcs in $X \setminus \{u_k\}$. ∎

CLAIM 3: $\gamma_m < \gamma_{m+1} < \cdots < \gamma_n$.

*Proof:* Assume that $k > m$ is such that $\gamma_{k-1} = \gamma_k < \gamma_{k+1} < \ldots < \gamma_n$. Then $u_k$ is a **triple point** in $X$, i.e., a point where three disjoint arcs in $X$ meet: $K_k$, and the graphs of $\xi_{k-1}|[0,s)$ and $\xi_k|[0,s)$. On the other hand, by Claim 2 and Lemma 9 there exist a line $\ell$ through $u_k$ and an $\varepsilon > 0$ such that every component of $U_\varepsilon(\ell) \cap X$ is homeomorphic to $\mathbb{R}$. Obviously, we have a contradiction. ∎

Observe that we may now assume that the arcs $K_{m+1}, \ldots, K_n$ are pairwise disjoint.

CLAIM 4: $m \geq 2$.

*Proof:* If $m \leq 1$ then according to Claim 3, $\gamma_2 < \ldots < \gamma_n$ and $u_i$ is a crossing of the vertical line $\{s\} \times \mathbb{R}$ with the arc $(\xi_i|[0,s)) \cup K_i$ for each $i \in \{2, \ldots, n\}$. By Lemma 9 we have that there is an $\varepsilon > 0$ such that every $\xi_i|(s-\varepsilon, s+\varepsilon)$ is continuous, violating the definition of $s$. ∎

CLAIM 5: $\gamma_m = \gamma_{m-1}$.

*Proof:* Assume that $\gamma_{m-1} < \gamma_m$ and draw a line $f(x) = \gamma_m + \mu(x - s)$ through $u_m$ with $\mu$ large enough so that $\mu > N$, $f(r) > t$, and $f(0) < \xi_1(0)$. Then this line $\ell$ cuts the graphs of $\xi_1|[0,s), \ldots, \xi_{m-1}|[0,s)$ and it cuts each of the arcs $K_{m+1}, \ldots, K_n$ for a total of $n-1$ crossings; see Figure 5.

By Lemma 9 there exists an $\varepsilon > 0$ such that every component of $U_\varepsilon(\ell) \cap X$ is a closed subset of $U_\varepsilon(\ell)$ that is homeomorphic to $\mathbb{R}$. This means that $u_m$ lies on a copy of $\mathbb{R}$ in $X$ and hence two arcs meet at $u_m$, one of which is the graph of $\xi_m|[0,s)$. So there is an arc $K \subset X$, disjoint from the graphs of every $\xi_i|[0,s)$, that contains $u_m$. Since $K$ can intersect the line $\{s\} \times \mathbb{R}$ in only finitely many points we may assume that $\pi_1(K \setminus \{u_m\}) \subset (s, \infty)$ or $\pi_1(K \setminus \{u_m\}) \subset [0,s)$. The first inclusion violates the definition of $m$ so consider the second option. Select a point $(a,b) \in K$ such that $a < s$. Then the vertical line $\{a\} \times \mathbb{R}$ intersects $X$ in $n+1$ distinct points: $(a,b), (a, \xi_1(a)), \ldots, (a, \xi_n(a))$. ∎

Consider the continuous function $h$: $[0,s) \to \mathbb{R}$ whose value is the slope of the secant line through $(x, \xi_{m-1}(x))$ and $u_m = u_{m-1}$, i.e.,

$$h(x) = \frac{\gamma_{m-1} - \xi_{m-1}(x)}{s - x}.$$

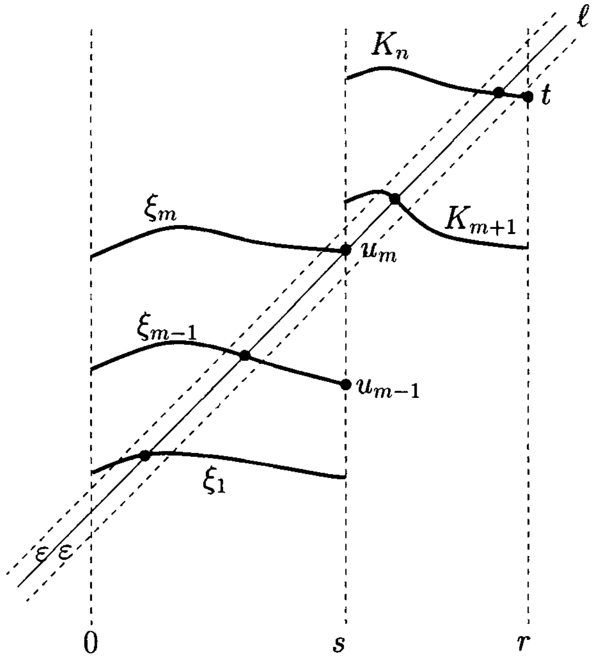


Figure 5

CLAIM 6: *$h$ is unbounded above.*

*Proof:* Let $B$ be such that $h(x) < B$ for each $x \in [0, s)$ and $B > N$. Using Claims 2 and 3 we can find a $\mu > B$ such that the line $\ell$ through $u_m$ that is the graph of the function $f(x) = \gamma_m + \mu(x - s)$ cuts $n - 1$ pairwise disjoint arcs $C_1, \ldots, C_{n-1}$ in $X \setminus \{u_m\}$. Since $\mu > h(x)$ and $\gamma_m = \gamma_{m-1}$ we have $f(x) < \xi_{m-1}(x) < \xi_m(x)$ for every $x \in [0, s)$. So we may assume that every $C_i$ is disjoint from the graphs of $\xi_{m-1}|[0, s)$ and $\xi_m|[0, s)$. It is obvious that we can find a line $\ell'$, parallel and slightly above $\ell$ such that $\ell$ meets every $C_i$ and the graphs of both $\xi_{m-1}|[0, s)$ and $\xi_m|[0, s)$; see Figure 6.

So $\ell'$ intersects $X$ in at least $n + 1$ points and the claim is proved because the slope of $\ell$, $\mu$, is greater than $N$. ∎

Select an increasing sequence $a_1, a_2, \ldots$ in $(0, s)$ such that $\lim_{j\to\infty} a_j = s$, $h(a_1) < h(a_2) < \cdots$, and $\lim_{j\to\infty} h(a_j) = \infty$. For each $j \in \mathbb{N}$ let $\ell_j$ be the line through the points $(a_j, \xi_{m-1}(a_j))$ and $u_{m-1} = u_m$, i.e., $\ell_j$ is the graph of the function

$$f_j(x) = \gamma_m + h(a_j)(x - s) = \xi_{m-1}(a_j) + h(a_j)(x - a_j).$$

Since $h(a_{j+1}) > h(a_j)$ we have that

$$\xi_{m-1}(a_{j-1}) = \gamma_m + h(a_{j+1})(a_{j+1} - s) < \gamma_m + h(a_j)(a_{j+1} - s) = f(a_{j+1})$$

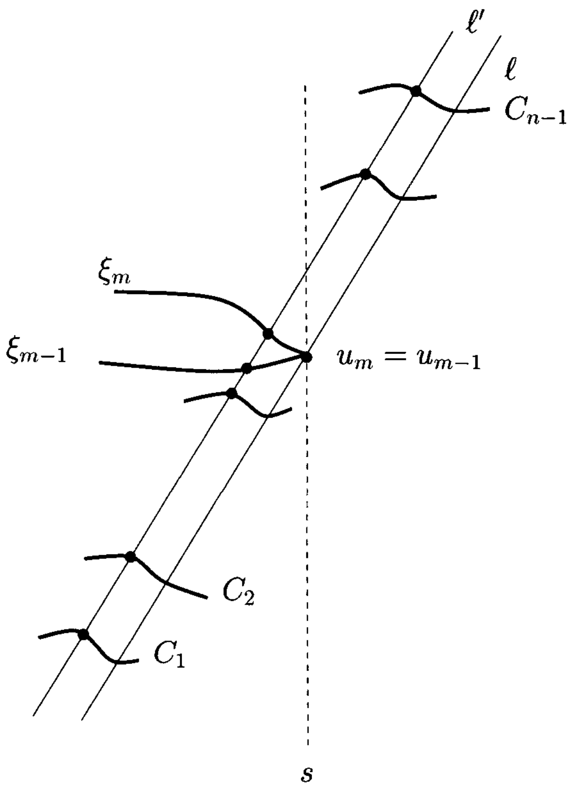


Figure 6

so $(a_{j+1}, \xi_{m-1}(a_{j+1}))$ lies below $\ell_j$. Let $v_j = (b_j, \xi_{m-1}(b_j))$ be a point on the graph of $\xi_{m-1}|(a_j, s)$ that lies below $\ell_j$ and has maximum distance towards $\ell_j$. The line $\ell'_j$ that is parallel to $\ell_j$ and passes through $v_j$ can be described by the function

$$g_j(x) = \xi_{m-1}(b_j) + h(a_j)(x - b_j).$$

Observe that $\lim_{j\to\infty} g_j(r) = \gamma_{m-1} + \infty(r - s) = \infty$ and $\lim_{j\to\infty} g_j(0) = \gamma_{m-1} - \infty \cdot s = -\infty$. Choose a $k \in \mathbb{N}$ large enough so that $g_k(r) > t$, $g_k(0) < \xi_1(0)$, and

$h(a_k) > N$. See Figure 7.

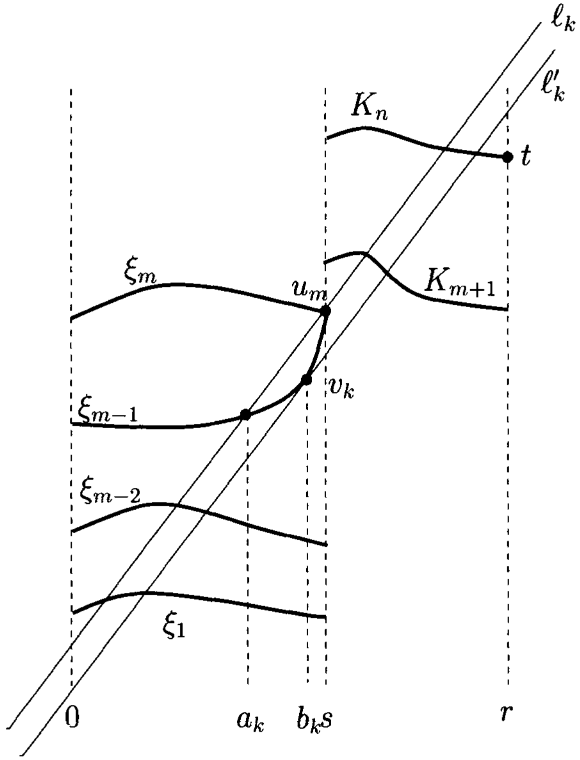


Figure 7

Since $g_k(0) < \xi_i(0)$ and $g_k(b_k) = \xi_{m-1}(b_k) > \xi_i(b_k)$ for every $i \in \{1, \ldots, m-2\}$ we have that $\ell'_k$ cuts the graphs of $\xi_1|[0, b_k], \ldots, \xi_{m-2}|[0, b_k]$. Since $g_k(s) < \gamma_m$ and $g_k(r) > t$, $\ell'_k$ also cuts $K_{m+1}, \ldots, K_n$. That is a total of $n-2$ crossings distinct from $v_k$. Let $C_1, \ldots, C_{n-2}$ be pairwise disjoint arcs in $X$ and let $w_1, \ldots, w_{n-2}$ be points on $\ell'_k$ such that for every $i \le n-2$, $w_i$ is a crossing of $\ell'_k$ with $C_i$ and $C_i$ is disjoint from the graph of $\xi_{m-1}|(a_k, s)$. Since the slope of $\ell'_k$ is greater than $N$ the line meets $X$ in $n$ points so there must be a point $w_0 = (c, g_k(c))$ on $\ell'_k$ that is distinct from $w_1, \ldots, w_{n-2}$ and $v_k$. If necessary, shorten a $C_i$ so that none of them contains $w_0$. In addition, choose an $\varepsilon > 0$ such that $[b_k - \varepsilon, b_k + \varepsilon] \subset (a_k, s)$ and $w_0$ does not lie on the graph of $\xi_{m-1}|[b_k - \varepsilon, b_k + \varepsilon]$. Just as in the proof of Lemma 8 we can find a slope $\mu$ close to $h(a_k)$ so that the line $y = g_k(c) + \mu(x - c)$, which is obtained by rotating $\ell'_k$ slightly about $w_0$, still cuts every $C_i$ and also the graphs of both $\xi_{m-1}|[b_k - \varepsilon, b_k)$ and $\xi_{m-1}|(b_k, b_k + \varepsilon]$.

If $c > b_k$ then we choose $\mu$ slightly smaller than $h(a_k)$ but greater than $N$ and if $c < b_k$ then $\mu > h(a_k)$; see Figure 8 for the case $c > b_k$.

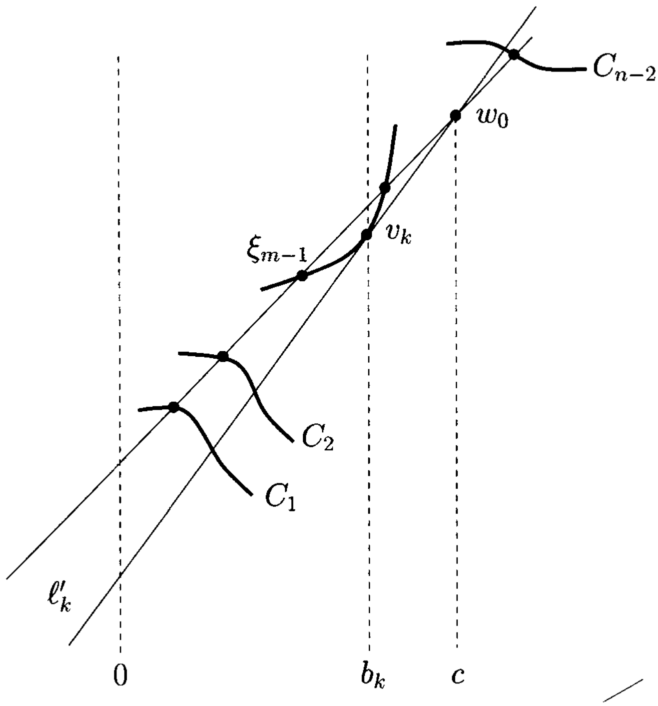


Figure 8

By rotating we have turned the point $v_k$ into two points of intersection so we have that $\ell'_k$ intersects $X$ in at least $n+1$ points. Note that this contradiction is obtained assuming only that $s$ exists so we conclude that every $\xi_i$ is continuous on the whole real line. Theorem 10 is proved. ∎

The First Structure Theorem follows immediately from the next result. Let us call a planar set $X$ **almost** an $n$-point set if there is a nowhere dense subset $Z$ of $\mathbb{P}$ such that every line $\ell$ with $\theta(\ell) \notin Z$ and $\ell \cap \overline{X} \neq \emptyset$ meets $X$ in precisely $n$ points.

THEOREM 11: *Assume that $X \subset \mathbb{R}^2$ is almost an $n$-point set. If there is a line that cuts $n-1$ pairwise disjoint subcontinua of $X$ then $X$ consists of $n$ parallel lines.*

*Proof:* Let $U$ be a dense open subset of $\mathbb{P}$ such that every line $\ell$ that meets $\overline{X}$ and with slope $\theta(\ell)$ in $U$ intersects $X$ in exactly $n$ points. If there is a line that cuts $n-1$ disjoint continua then this property is preserved under small rotations of the line; see Figure 1. So we may assume that the line $\ell$ that cuts $n-1$ subcontinua of $X$ has the property $\theta(\ell) \in U$. Choose an $xy$-coordinate system such that $\ell$ is the $y$-axis. According to Theorem 10 there exist continuous functions $\xi_1 < \cdots < \xi_n : \mathbb{R} \to \mathbb{R}$ such that $X$ is the union of their graphs $y = \xi_i(x)$. Note that every point of $X$ is an arc-point.

It suffices to show that every $\xi_i$ is a linear function. Assume that some $\xi_i$ is not linear so we can find real numbers $a < b < c$ such that $(a, \xi_i(a))$, $(b, \xi_i(b))$, and $(c, \xi_i(c))$ are not collinear. Then we can find a line $L$ through $(a, \xi_i(a))$ that separates $(b, \xi_i(b))$ from $(c, \xi_i(c))$. Because $U$ is a dense set of directions we can arrange that $\theta(L) \in U$. Note that any line through $(a, \xi_i(a))$ with angle of inclination in $U$ intersects $X$ in $n-1$ other points, all of which are arc-points of $X \setminus \{(a, \xi_i(a))\}$. So Lemma 8 applies and we may assume that $L$ cuts $n-1$ pairwise disjoint arcs in $X$ while retaining the property $\theta(L) \in U$. In addition, since the rotation can be chosen arbitrarily small, we can arrange that $L$ still separates $(b, \xi_i(b))$ from $(c, \xi_i(c))$. Note that $L$ intersects a component of $X$, namely the graph of $\xi_i$, in at least two points, $(a, \xi_i(a))$ and somewhere between $b$ and $c$.

Select an $x'y'$-coordinate system such that $L$ is the $y'$-axis. According to Theorem 10 every component of $X$ is the graph of a function $y' = \psi(x')$. Since the $y'$-axis intersects a component twice we have a contradiction. ∎

We derive the Second Structure Theorem from the First by combining the following lemma with Kulesza's proof [12] that two-point sets are zero-dimensional.

If $X$ is a space then we define $X^* = \{u \in X : \operatorname{ind}_u X > 0\}$. If $u \in \mathbb{R}^2$ and $V \subset \mathbb{P}$ then we define

$$\operatorname{sect}_u V = \{v \in \mathbb{R}^2 \setminus \{u\} : \theta(L(u,v)) \in V\}.$$

Note that $\operatorname{sect}_u V$ is open if $V$ is open. If $V \subset \mathbb{P}$ then we call a planar set $X$ a **partial $n$-point set rel** $V$ if every line $\ell$ with $\theta(\ell) \in V$ intersects $X$ in at most $n$ points. The concept of an $n$-point set is of course meaningless if $n < 2$. However, for partial $n$-point sets we do allow $n$ to assume the values 0 and 1.

LEMMA 12: *If $V$ is a nonempty open subset of $\mathbb{P}$ and $X$ is a planar set such that $X^*$ is a partial one-point set rel $V$ then either* $\operatorname{ind} X \leq 0$ *or $X$ contains an arc.*

*Proof:* Let $V$ be a nonempty open subset of $\mathbb{P}$, let $\operatorname{ind} X > 0$, and let $X^*$ be a partial one-point set rel $V$. We may assume without loss of generality that $\pi/2 \in V$ and that $X$ is bounded. Let $B$ be the compactum that is the closure of $X^*$ in $\mathbb{R}^2$ and note that $\operatorname{ind}(B \cap X) > 0$ by [10, Corollary 1.5.5]. We show that $g = \pi_1|B$ is a one-to-one function. Choose an $\varepsilon \in (0, \pi/2)$ with $(\pi/2 - \varepsilon, \pi/2 + \varepsilon) \subset V$. Observe that if $(x_1, y_1)$ and $(x_2, y_2)$ are points in $X^*$ then $|y_2 - y_1| \leq (\cot \varepsilon)|x_2 - x_1|$. Let $(c, d)$ and $(c, d')$ be points of $B$ and select sequences $(x_i, y_i)$ and $(x_i', y_i')$ in $X^*$ that converge to $(c, d)$ respectively $(c, d')$. Then $d = d'$ because

$$|d' - d| = \lim_{i \to \infty} |y_i' - y_i| \leq (\cot \varepsilon) \lim_{i \to \infty} |x_i' - x_i| = (\cot \varepsilon)|c - c| = 0.$$

Since $B$ is compact we have that $g\colon B \to \pi_1(B)$ is a homeomorphism so $g(B \cap X)$ is a one-dimensional subset of $\mathbb{R}$. Let $[a, b]$ be a non-degenerate interval in $g(B \cap X)$ and observe that $g^{-1}([a, b])$ is an arc that is contained in $X$. ∎

Inspection of Kulesza's proof [12] that every one-dimensional partial two-point set contains arcs shows that it also supports the following result, cf. [2, §4.3].

LEMMA 13: *Let $V$ be open in $\mathbb{P}$ and let $X \subset \mathbb{R}^2$ be a partial two-point set rel $V$. If $u$ is a point in $X^*$ then either* $\operatorname{ind}(X \cap \operatorname{sect}_u V) \leq 0$ *or $X \cap \operatorname{sect}_u V$ contains an arc.*

If we combine the following result with Structure Theorem 1 then we have Structure Theorem 2.

THEOREM 14: *Let $\ell$ be a line and let $V$ be a neighbourhood of $\theta(\ell)$ in $\mathbb{P}$. Let $X$ be a partial $n$-point set rel $V$ that contains $n-2$ pairwise disjoint continua $C_1, \ldots, C_{n-2}$ that are all cut by $\ell$. If there is a point $u \in X^* \cap \ell \setminus \bigcup_{i=1}^{n-2} C_i$ then there are an arc $C_{n-1} \subset X \setminus \bigcup_{i=1}^{n-2} C_i$ and a line $\ell'$, parallel to $\ell$, that cuts $C_1, \ldots, C_{n-1}$.*

*Proof:* It is easily seen that we can find an open neighbourhood $U \subset \mathbb{R}^2 \setminus \bigcup_{i=1}^{n-2} C_i$ of $u$ and an open interval $W \subset V$ around $\theta(\ell)$ such that any $\ell'$ that meets $U$ and with $\theta(\ell') \in W$ cuts each of the continua $C_1, \ldots, C_{n-2}$. Consequently, $U \cap X$ is a partial two-point set rel $W$.

If $U \cap X$ contains an arc $C_{n-1}$ then, since $U \cap X$ is a partial two-point set rel $W$, the arc $C_{n-1}$ is not contained in any line that is parallel to $\ell$. We can obviously find a line $\ell'$, parallel to $\ell$, that cuts $C_{n-1}$. Then $\ell'$ also cuts all the other $C_i$'s and the proof is finished.

So we may assume that $U \cap X$ contains no arcs. Note that $u \in (U \cap X)^*$. If we apply Lemma 12 to $U \cap X$ then we find that $(U \cap X)^*$ is no partial one-point set rel $W$ so there are two points $v$ and $w$ in $(U \cap X)^*$ such that $\theta(L(v,w)) \in W$. This result contradicts Lemma 13. ∎

## 3. Extension theorems

Every subset of an $n$-point set is a partial $n$-point set but not every partial $n$-point set is contained in some $n$-point set. In this section we address the extension problem, that is, the question under what conditions a partial $n$-point set is contained in some $n$-point set. Note that the structure theorems are "negative extension theorems" because, for instance, Theorem 1 states that whenever a planar compactum $C$ consists of $n-1$ pairwise disjoint arcs that are all cut by some line then $C$ is not extendible to an $n$-point set. The forms that positive extension theorems can take are of course limited by the structure theorems. Let us consider some illustrations.

*Example 1:* Let $n = 2k$ be an even number greater than 2. Let $C$ be the union of $k-1$ concentric circles. Since $C$ is obviously a partial $(n-2)$-point set we have according to [4, Theorem 5.2] that $C$ is contained in some $n$-point set. Obviously, if $u$ is an arbitrary point in the plane then a line through $u$ and the centre of the circles cuts $n-2$ pairwise disjoint arcs in $C$. Consequently, by the Second Structure Theorem any $n$-point set $X$ that contains $C$ has the property $\operatorname{ind}(X \setminus C) = 0$.

*Example 2:* Let $n = 2k+1$ be an odd integer greater than 3. Let the compactum $C$ be the union of $k-1$ circles that are tangent to a line $\ell$ at the origin together with a circular arc $C_0$ with its endpoints on $\ell$ as described by Figure 9.

Observe that $C$ is a partial $(2k-1)$-point set so $C$ is extendible to an $n$-point set. Let $X$ be such an extension of $C$. If $u \notin \ell$ then we can draw a line $\ell'$ through $u$ and a point $v$ inside $C_1$ that is close enough to $\mathbf{0}$ so that $\ell'$ separates the endpoints of $C_0$. So $\ell'$ cuts $C_0$ and meets each circle in two points. According to the Second Structure Theorem we have $\operatorname{ind}(X \setminus (C \cup \ell)) = 0$. Since $X \cap \ell$ is finite we may conclude that $\operatorname{ind}(X \setminus C) = 0$ by [10, Corollary 1.5.6].

These examples suggest that extension theorems should have a form similar to the following naive conjecture: *if $C$ is a compact partial $(n-2)$-point set and $Z$ is a zero-dimensional compactum such that $C \cup Z$ is a partial $n$-point set then $C \cup Z$ is extendible to an $n$-point set.*

This conjecture was shown to be false in the case $n = 2$ by Dougherty [9] and Dijkstra and van Mill [7]. Later in this section (Propositions 21 and 22) we show that for every $n \geq 2$ there exist Cantor sets that are partial $n$-point sets but fail to be extendible to $n$-point sets.

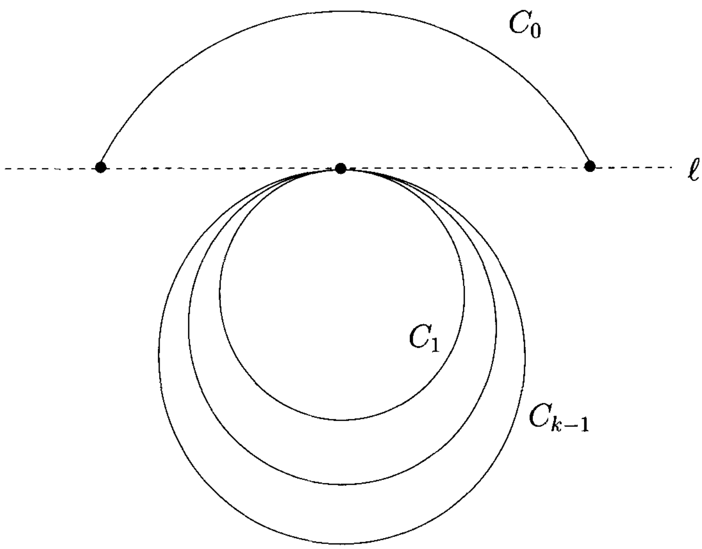


Figure 9

We will present three positive extension theorems in this section, Theorems 15, 17, and 19, that have the same basic form as the conjecture: all three contain the premises that $Y$ is a partial $n$-point set, $Z$ is a subset of $Y$ such that $C = Y \setminus Z$ is a partial $(n-2)$-point set in tandem with a condition that forces $Z$ to be zero-dimensional.

Theorem 5.2 in [4] states that every partial $(n-2)$-point set can be extended to an $n$-point set. We observe that the following improvement can be obtained by a slight modification of the proof given in [4]. Let $\mathfrak{c}$ denote $2^{\aleph_0}$.

THEOREM 15: *Let $n \geq 2$, let $Y$ be a partial $n$-point set, and let $Z$ be a subset of $Y$. If $|Z| < \mathfrak{c}$ and $Y \setminus Z$ is a partial $(n-2)$-point set then $Y$ is contained in some $n$-point set.*

*Proof:* In the proof of [4, Theorem 5.2] we start the induction with $E_0 = Z$ instead of $E_0 = \emptyset$. ∎

*Example 3:* Let $n \geq 4$ and let $Y$ be the union of $n-2$ disjoint arcs as shown in Figure 10. Note that the arcs are arranged in such a way that any line that intersects a single arc twice does not meet any of the other arcs. Consequently, $Y$ is a partial $(n-2)$-point set, provided that $n \geq 4$. So by Theorem 15 the set $Y$ is contained in some $n$-point set $X$. Since there are lines that cut all $n-2$ arcs we have that the count of $n-2$ continua in the Structure Theorem cannot be lowered.

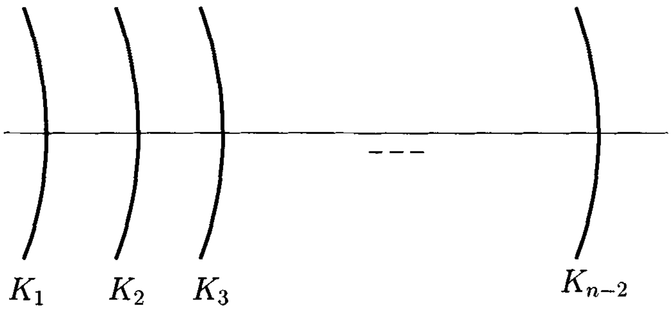


Figure 10

*Example 4:* Let $n \geq 4$, let $\ell$ be the $x$-axis, and let $K_1, K_2, \ldots, K_{n-1}$ be disjoint arcs as in Figure 11.

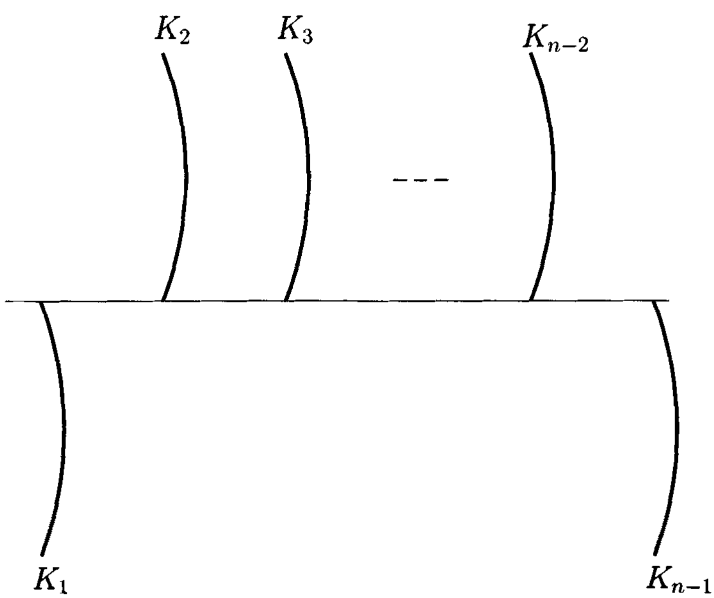


Figure 11

Note that $\ell$ meets but fails to cut every $K_i$. Put $Y = \bigcup_{i=1}^{n-1} K_i$ and $Z = \ell \cap Y$ and observe that $|Z| = n - 1 < \mathfrak{c}$ and that $Y \setminus Z$ is a partial $(n-2)$-point set. By Theorem 15 the set $Y$ is extendible to an $n$-point set $X$. This example shows that in the Structure Theorem the term "cuts" cannot be replaced by "meets".

In connection with the same issue we obtain the following corollary to the Structure Theorem:

COROLLARY 16: *If $X$ is a four-point set then every line meets at most three disjoint subcontinua of $X$.*

*Proof:* Let $X$ be a 4-point set and let $\ell$ be a line that meets four pairwise disjoint subcontinua of $X$. According to [17] we may assume that the continua are arcs. Since $X$ is a 4-point set every arc intersects $\ell$ in only one point. Choose a coordinate system for the plane and choose names $C_1, C_2, C_3, C_4$ for the arcs such that $\ell$ is the $x$-axis, $C_i \cap \ell = \{(x_i, 0)\}$ for $i \in \{1,2,3,4\}$, and $x_1 < 0 < x_2 < x_3 < x_4$. If necessary, shorten each $C_i$ so that $(x_i, 0)$ is an endpoint. Consequently, each arc lies either in the upper or in the lower halfplane. By symmetry we may assume that $C_1$ lies in the lower halfplane. If three of the arcs lie on the same side of $\ell$ then by shifting $\ell$ slightly into that halfplane we obtain that the resulting parallel line cuts the three arcs, in violation of the Structure Theorem.

So we may assume that the upper halfplane contains two of the arcs, say $C_i$ and $C_j$. If $u$ is a point in the plane other than the origin then $\alpha(u) \in \mathbb{P}$ denotes the angle of inclination of the line through the origin and $u$. Note that the image under $\alpha$ of each of the arcs $C_1$, $C_i$, and $C_j$ has the form $[0, \varepsilon]$, with $0 < \varepsilon < \pi$. Select a $\delta$ in the interior of the smallest of the three intervals and note that the line through the origin with angle of inclination $\delta$ cuts $C_1$, $C_i$, and $C_j$. We have obtained a contradiction with the Structure Theorem. ∎

*Example 5:* For every $n \geq 5$ there exist an $n$-point set $X$ and a line $\ell$ that meets $X$ in $n$ pairwise disjoint arcs, showing that Corollary 16 does not generalize to higher $n$.

Let $Y$ be the union of the arcs $K_1, K_2, \ldots, K_n$ that are positioned in an alternating pattern around a line $\ell$; Figure 12 shows the situation for odd $n$. Put $Z = Y \cap \ell$. Observe that if $\ell'$ is a line distinct from $\ell$ and $n$ is odd then $\ell'$ meets $Y$ in at most $(n+1)/2$ points and that $(n+1)/2 \leq n-2$ because $n \geq 5$. If $n$ is even then $|Y \cap \ell'| \leq (n+2)/2 \leq n-2$ because $n \geq 6$. So $Y \setminus Z$ is a partial $(n-2)$-point set and $|Z| = n$ which means that $Y$ is extendible to some $n$-point set by Theorem 15.

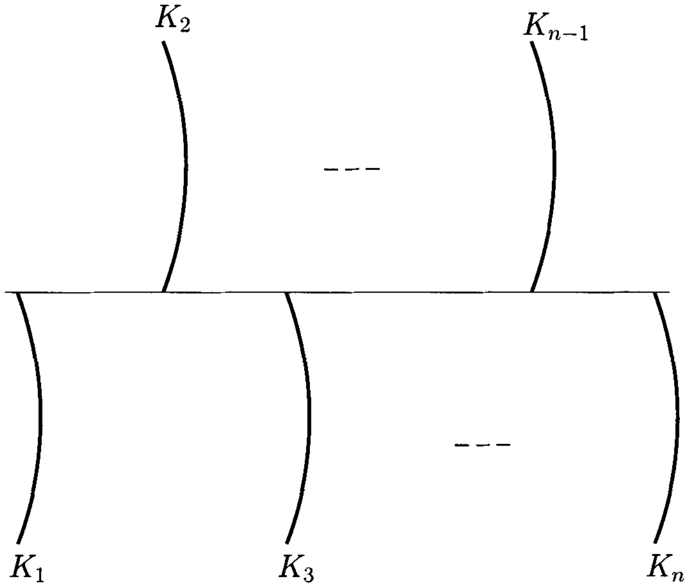


Figure 12

It was observed in [4] that no $(n-1)$-point set is extendible to an $n$-point set. Using Martin's Axiom we obtain the following result about extending partial $(n-1)$-point sets to $n$-point sets.

THEOREM 17 (MA): *Let $n \geq 3$, let $Y$ be a $\sigma$-compact partial $(n-1)$-point set, and let $Z$ be a $\sigma$-compact subset of $Y$. If $\operatorname{ind} Z \leq 0$ and $Y \setminus Z$ is a partial $(n-2)$-point set then $Y$ is contained in some $n$-point set.*

*Proof:* Let $Y$, $Z$ and $Y \setminus Z$ be as described in the theorem. Let $\{\ell_\alpha \colon \alpha < \mathfrak{c}\}$ enumerate the lines in the plane. We construct by transfinite recursion a nondecreasing sequence $\{E_\alpha\}_{\alpha<\mathfrak{c}}$ of subsets of $\mathbb{R}^2 \setminus Y$ with the following hypotheses:

(1) $|E_\alpha| \leq |\alpha| + \aleph_0$,

(2) $Y \cup E_\alpha$ is a partial $n$-point set,

(3) $|(Y \cup E_{\alpha+1}) \cap \ell_\alpha| = n$.

Observe that these properties imply that $Y \cup \bigcup_{\alpha<\mathfrak{c}} E_\alpha$ is an $n$-point set. It remains to perform the construction.

Put $E_0 = \emptyset$ and if $\lambda < \mathfrak{c}$ is a limit ordinal then $E_\lambda = \bigcup_{\alpha<\lambda} E_\alpha$. Let $\alpha$ be a fixed ordinal number below $\mathfrak{c}$ and consider $E_\alpha$ and $\ell_\alpha$. Let $\mathcal{L}$ stand for the collection of all lines that meet $Y \cup E_\alpha$ in precisely $n$ points. If $\ell_\alpha \in \mathcal{L}$ then we put $E_{\alpha+1} = E_\alpha$. In all these cases the induction hypotheses are trivially satisfied.

Assume now that $\ell_\alpha \notin \mathcal{L}$ so every $\ell \in \mathcal{L}$ has at most one point in common with $\ell_\alpha$. The set $\ell_\alpha \cap \bigcup \mathcal{L}$ is the set of "forbidden" points, because if we add any of these points to $E_\alpha$ in order to satisfy statement (3) then we will violate hypothesis (2) for $\alpha + 1$. We shall prove that $\ell_\alpha \cap \bigcup \mathcal{L}$ has a dense complement in $\ell_\alpha$, giving us many "good" points to choose from.

Let $u$ be a fixed point of $E_\alpha \setminus \ell_\alpha$. If $A$ is a subset of the plane that does not contain $u$ then $\mathfrak{C}_u(A)$ denotes the cone over $A$ with respect to the vertex $u$, i.e., the union of all lines through $u$ that intersect $A$. Observe that if $A$ is compact then $\mathfrak{C}_u(A)$ is closed in $\mathbb{R}^2$. Let $\ell_u$ be the line through $u$ that is parallel to $\ell_\alpha$. Since $Y \setminus \ell_u$ is $\sigma$-compact and second countable we can find compact sets $F_i^j \subset Y \setminus \ell_u$, for $j \in \mathbb{N}$ and $i \in \{1, 2, \ldots, n-1\}$, such that for each $j \in \mathbb{N}$ the collection $\{F_1^j, F_2^j, \ldots, F_{n-1}^j\}$ is pairwise disjoint and for each sequence $v_1, v_2, \ldots, v_{n-1}$ of distinct points from $Y \setminus \ell_u$ there is a $j \in \mathbb{N}$ with $v_i \in F_i^j$ for every $i$.

Let $j$ be an arbitrary natural number and define

$$Z_u^j = Z \cap F_1^j \cap \bigcap_{i=2}^{n-1} \mathfrak{C}_u(F_i^j).$$

Observe that $Z_u^j$ is a $\sigma$-compact zero-dimensional set. Define the radial projection $p_u^j \colon Z_u^j \to \ell_\alpha$ by letting $p_u^j(z)$ be the point of intersection of the line $L(u, z)$ through $u$ and $z$ with $\ell_\alpha$. Since $z \in F_1^j \subset Y \setminus \ell_u$ we have that $z \neq u$ and that $L(u, z)$ is not parallel to $\ell_\alpha$. So $p_u^j$ is well-defined which automatically implies continuity.

We now prove that $p_u^j$ is one-to-one. Let $z$ and $z'$ be two points of $Z_u^j$ such that $p_u^j(z) = p_u^j(z')$. Since $u \notin \ell_\alpha$ we have $L(u, z) = L(u, p(z)) = L(u, p(z')) = L(u, z')$. Since $z$ and $z'$ are points of $Z_\alpha^j(u)$ we have that they both belong to $F_1^j$ and that $z \in \bigcap_{i=2}^{n-1} \mathfrak{C}_u(F_i^j)$ which means that we can find a point $v_i \in F_i^j \cap L(u, z)$ for every $i \in \{2, 3, \ldots, n-1\}$. Because of pairwise disjointness of the $F_i^j$'s we have that both $\{z, v_2, v_3, \ldots, v_{n-1}\}$ and $\{z', v_2, v_3, \ldots, v_{n-1}\}$ have precisely $n-1$ elements all of which lie in $Y \cap L(u, z) = Y \cap L(u, z')$. Since $Y$ is a partial $(n-1)$-point set we must have that $z = z'$.

So we may conclude that $B_u^j = p_u^j(Z_u^j)$ is a zero-dimensional $\sigma$-compactum and hence a first category set in $\ell_\alpha$. Since $|E_\alpha| < \mathfrak{c}$ we have with Martin's Axiom that the set

$$B = ((Y \cup E_\alpha) \cap \ell_\alpha) \cup \bigcup_{u \in E_\alpha \setminus \ell_\alpha} \left( \bigcup_{v \in E_\alpha \setminus \{u\}} (L(u, v) \cap \ell_\alpha) \cup \bigcup_{j=1}^{\infty} B_u^j \right)$$

has a complement in $\ell_\alpha$ that is dense. So the set $\ell_\alpha \setminus B$ is infinite and we form $E_{\alpha+1}$ by picking $n - |(Y \cup E_\alpha) \cap \ell_\alpha|$ points from $\ell_\alpha \setminus B$ and adding them to $E_\alpha$. Observe that $E_{\alpha+1} \cap Y = \emptyset$, that $|E_{\alpha+1}| \le |E_\alpha| + n \le |\alpha| + \aleph_0$, and that $|(Y \cup E_{\alpha+1}) \cap \ell_\alpha| = n$.

It remains to verify hypothesis (2), i.e., that $Y \cup E_{\alpha+1}$ is a partial $n$-point set. Assume that there is a line $\ell$ such that $|(Y \cup E_{\alpha+1}) \cap \ell| \ge n+1$. Then $\ell \ne \ell_\alpha$ and since $Y \cup E_\alpha$ is a partial $n$-point set we have that there is a $w \in E_{\alpha+1} \setminus E_\alpha$ such that $\ell \cap \ell_\alpha = \{w\}$. So $\ell$ contains precisely $n$ points from $Y \cup E_\alpha$ and hence $\ell \in \mathcal{L}$. We show that $B$ contains the forbidden point $w$, contradicting the property $B \cap E_{\alpha+1} \setminus E_\alpha = \emptyset$.

Since $|(Y \cup E_\alpha) \cap \ell| = n$ and $Y$ is a partial $(n-1)$-point set we can find a point $u \in \ell \cap E_\alpha$. Note that $u \notin \ell_\alpha$. If there is another point $v \in \ell \cap E_\alpha$ then $w \in L(u,v) \cap \ell_\alpha \subset B$.

If $u$ is the only point in $\ell \cap E_\alpha$ then $|Y \cap \ell| = n-1$ which implies that there is a point $z \in \ell \cap Z$ because $Y \setminus Z$ is a partial $(n-2)$-point set. The other $n-2$ points in $Y \cap \ell$ will be called $v_2, v_3, \ldots, v_{n-1}$. Note that $\{z, v_2, v_3, \ldots, v_{n-1}\} \subset Y \setminus \ell_u$ because $\ell_\alpha$ and $\ell$ are not parallel. Then we can find a $j \in \mathbb{N}$ such that $z \in F_1^j$ and $v_i \in F_i^j$ for $2 \le i \le n-1$. So $z \in Z_u^j$ and $w = p_u^j(z) \in B_u^j \subset B$. ∎

The next example shows that the $\sigma$-compactness of $Z$ in Theorem 17 is essential.

*Example 6:* Let $n \ge 4$ and let $K_1, K_2, \ldots, K_{n-1}$ be $n-1$ arcs as in Figure 13 and let $Q_i$ be a countable dense subset of $K_i$ for each $1 \le i \le n-2$. Let $\mathcal{L} = \{L(u,v)\colon u \ne v \text{ and } u, v \in \bigcup_{i=1}^{n-2} Q_i\}$ and define the countable set $D = K_{n-1} \cap \bigcup \mathcal{L}$. Let $Q_{n-1}$ be a countable dense subset of $K_{n-1} \setminus D$.

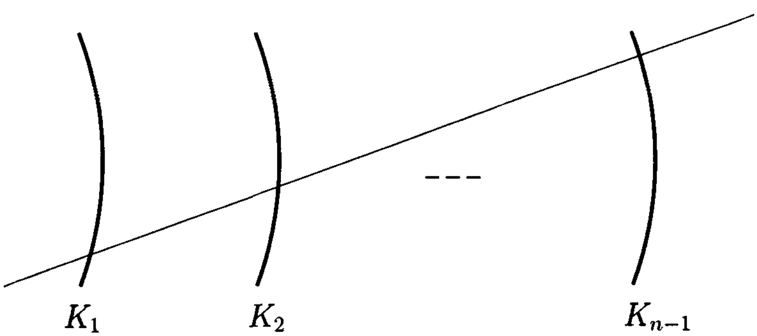


Figure 13

Define $Y = \bigcup_{i=1}^{n-1} K_i$ and $Z = Y \setminus \bigcup_{i=1}^{n-1} Q_i$. Then $Z$ is a zero-dimensional

$G_\delta$-subset of the compact partial $(n-1)$-point set $Y$. We check that $Y \setminus Z$ is a partial $(n-2)$-point set. Suppose that there is a line $\ell$ that intersects $Y \setminus Z$ in $n-1 \geq 3$ points, which means that $\ell$ intersects each $Q_i$ for $1 \leq i \leq n-1$. Since the arcs are arranged in such a way that every line that intersects one of the $K_i$'s in two points does not meet any other arc we have that $\ell$ meets every $K_i$ in at most one point and hence $\ell$ must meet $\bigcup_{i=1}^{n-1} Q_i$ in at least two points. So $\ell \in \mathcal{L}$ and consequently $\ell$ is disjoint from $Q_{n-1}$. So $\ell$ meets $\bigcup_{i=1}^{n-2} K_i$ in $n-1$ points, which is not possible. Hence $Y \setminus Z$ is a partial $(n-2)$-point set.

It follows from the Structure Theorem that $Y$ cannot be extended to an $n$-point set.

COROLLARY 18 (MA): *A $\sigma$-compact partial two-point set $Y$ is extendible to a three-point set if and only if* $\operatorname{ind} Y \leq 0$.

*Proof:* Let $Y$ be a $\sigma$-compact partial two-point set in the plane and suppose that $\operatorname{ind} Y \leq 0$. Then by Theorem 17 $A$ is extendible to a three-point set.

Now let $Y$ be a $\sigma$-compact subset of a three-point set. According to [4, Theorem 4.4] $Y$ cannot contain nontrivial continua. By $\sigma$-compactness we have $\operatorname{ind} Y \leq 0$. ∎

To see that $\sigma$-compactness is necessary in the corollary let $X$ be a two-point set. Then according to [12] $X$ is zero-dimensional and by [4, Theorem 5.1] not extendible to a three-point set.

Let $\mathcal{H}^1$ denote the one-dimensional Hausdorff measure with respect to the standard Euclidean metric. It was proved under MA in [6] that whenever $Z$ is a $\sigma$-compact partial two-point set with $\mathcal{H}^1(Z \times Z) = 0$ then $Z$ is contained in some two-point set. We present a generalization of that result.

THEOREM 19 (MA): *Let $n \geq 2$, let $Y$ be a partial $n$-point set, and let $Z$ a subset of $Y$. If $\mathcal{H}^1(Z \times Z) = 0$ and $Y \setminus Z$ is a partial $(n-2)$-point set then $Y$ is contained in some $n$-point set.*

*Proof:* The beginning of the proof is identical to the first three paragraphs of the proof of Theorem 17. So we will restrict ourselves to describing the step from $\alpha$ to $\alpha+1$ of the recursion under the assumption $\ell_\alpha \notin \mathcal{L} = \{\ell\colon |(Y \cup E_\alpha) \cap \ell| = n\}$. Let $\lambda$ denote the Lebesgue measure on $\ell_\alpha$.

Let $u$ be an arbitrary point in $E_\alpha \setminus \ell_\alpha$. Let $\ell_u$ be the line through $u$ that is parallel to $\ell_\alpha$. Then $\mathbb{R}^2 \setminus \ell_u$ is an open subset of $\mathbb{R}^2$ and so we can write it as a countable union of compact sets $C_1, C_2, \ldots$. Define $\varphi_u\colon \mathbb{R}^2 \setminus \ell_u \to \ell_\alpha$ by $\{\varphi_u(z)\} = L(u,z) \cap \ell_\alpha$. Since $\varphi_u$ is a rational function and $C_i$ is compact the restriction $\varphi_u|C_i$ is Lipschitz for each $i \in \mathbb{N}$. Recall that the image of a set with measure zero under a Lipschitz map has again measure zero and that $\mathcal{H}^1$ on $\ell_\alpha$ is the Lebesgue outer measure. Since $\mathcal{H}^1(Z) \leq \mathcal{H}^1(Z \times Z) = 0$ we have $\mathcal{H}^1(\varphi_u(Z \cap C_i)) = \lambda(\varphi_u(Z \cap C_i)) = 0$ for each $i \in \mathbb{N}$. By $\sigma$-additivity of $\lambda$ we have $\lambda(\varphi_u(Z \setminus \ell_u)) = 0$.

Put $\Delta = \{(u,v) \in \mathbb{R}^2 \times \mathbb{R}^2 \colon v \in \ell_u\}$ and note that $\mathbb{R}^4 \setminus \Delta$ is an open subset of $\mathbb{R}^4$ on which we may define the rational function $f\colon \mathbb{R}^4 \setminus \Delta \to \ell_\alpha$ by putting $\{f(z,z')\} = L(z,z') \cap \ell_\alpha$. By the same Lipschitz argument as in the previous paragraph we have that $\mathcal{H}^1(Z \times Z) = 0$ implies $\lambda(f((Z \times Z) \setminus \Delta)) = 0$.

Note that the set

$$B = ((Y \cup E_\alpha) \cap \ell_\alpha) \cup f((Z \times Z) \setminus \Delta) \\ \cup \bigcup_{u \in E_\alpha \setminus \ell_\alpha} \Big(\varphi_u(Z \setminus \ell_u) \cup \bigcup_{v \in E_\alpha \setminus \{u\}} (L(u,v) \cap \ell_\alpha)\Big)$$

is a subset of $\ell_\alpha$ that is a union of finite sets, namely $(Y \cup E_\alpha) \cap \ell_\alpha$ and $L(u,v) \cap \ell_\alpha$, and of sets with Lebesgue measure zero, namely $f((Z \times Z) \setminus \Delta)$ and $\varphi_u(Z \setminus \ell_u)$. Since $|E_\alpha| < \mathfrak{c}$, Martin's Axiom guarantees that $\lambda(B) = 0$; see Rudin [18, Theorem 15]. So we can find a sufficient number of points in $\ell_\alpha \setminus B$ to add to $E_\alpha$ so that $|(Y \cup E_{\alpha+1}) \cap \ell_\alpha| = n$.

Again, the only part that requires verification is that $Y \cup E_{\alpha+1}$ is a partial $n$-point set. Assume that there is a line $\ell$ such that $|(Y \cup E_{\alpha+1}) \cap \ell| \geq n+1$. Then $\ell \neq \ell_\alpha$ and since $Y \cup E_\alpha$ is a partial $n$-point set we have that there is a $w \in E_{\alpha+1} \setminus E_\alpha$ such that $\ell \cap \ell_\alpha = \{w\}$. So $\ell$ contains precisely $n$ points from $Y \cup E_\alpha$ and hence $\ell \in \mathcal{L}$. We show that $B$ contains the forbidden point $w$, contradicting the property $B \cap E_{\alpha+1} \setminus E_\alpha = \emptyset$.

Since $|(Y \cup E_\alpha) \cap \ell| = n$ and $Y \setminus Z$ is a partial $(n-2)$-point set we can find two distinct points $u, v \in (Z \cup E_\alpha) \cap \ell$. Neither $u$ nor $v$ are in $\ell_\alpha$ because $w$ is the only point that $\ell_\alpha$ and $\ell$ have in common. If $u$ and $v$ are both in $E_\alpha$ then $w \in L(u,v) \subset B$. If $u$ and $v$ are both in $Z$ then $(u,v) \in (Z \times Z) \setminus \Delta$ because $\ell$ intersects $\ell_\alpha$. So in this case $w = f(u,v) \in B$. If $u \in E_\alpha$ and $v \in Z$ then $v \in Z \setminus \ell_u$ and hence $w = \varphi_u(v) \in B$. ∎

We now present a generalization of §2 in [6]. Let $D$ be a compact convex set in the plane that has the origin as interior point. We define the continuous maps $\alpha_D, \beta_D\colon \operatorname{int} D \setminus \{\mathbf{0}\} \to \partial D$ as follows: if $u \in \operatorname{int} D \setminus \{\mathbf{0}\}$ let $\ell_u$ be the line through $u$ that is perpendicular to the ray from $\mathbf{0}$ to $u$. The intersection of $\ell_u$ with $D$ is a compact interval on $\ell_u$ that has $u$ as an interior point. Denote the endpoints of the interval by $\alpha_D(u)$ respectively $\beta_D(u)$ in such a way that the direction of the vector from $\beta_D(u)$ to $\alpha_D(u)$ is obtained by rotating the vector from $\mathbf{0}$ to $u$ counterclockwise over 90 degrees; see Figure 14.

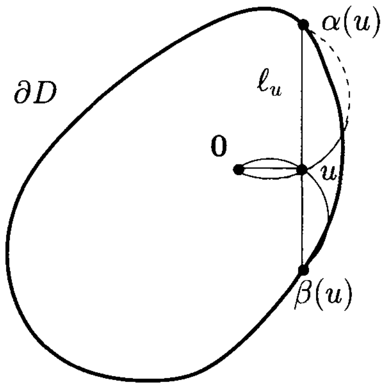


Figure 14

Observe that if $v \in \ell_u$ then $\mathbf{0}$, $u$, and $v$ lie on a circle that has the line segment from $\mathbf{0}$ to $v$ as a diameter. Consequently, the fibres of $\alpha_D$ and $\beta_D$ are subsets of semicircles that have $\mathbf{0}$ and a point on $\partial D$ as endpoints.

If $X \subset \mathbb{R}^2$ then $\mathcal{K}(X)$ denotes the space of non-empty compacta in $X$ equipped with the Hausdorff metric.

THEOREM 20: *Let $D_1, D_2, \ldots$ be a sequence of compact convex sets in the plane and let, for each $i \in \mathbb{N}$, $G_i$ be a dense $G_\delta$-subset of $\mathcal{K}(\partial D_i)$. Assume that $m = \inf\{\|u\|\colon u \notin \bigcap_{i=1}^{\infty} D_i\}$ and $M = \sup\{\|u\|\colon u \in \bigcup_{i=1}^{\infty} D_i\}$ are nonzero respectively finite. If $a$, $b$, and $c$ are real numbers with $0 < a < b < m \leq M < c$ then there exists a compact set $K \subset \{u \in \mathbb{R}^2\colon a \leq \|u\| \leq b\}$ such that for each $i \in \mathbb{N}$ both $\alpha_{D_i}(K)$ and $\beta_{D_i}(K)$ are elements of $G_i$ and moreover every semicircle $C$ that has as endpoints the origin and a point $u$ with $\|u\| \geq c$ meets $K$ in some point $v$.*

Observe that by elementary geometry the line $L(u, v)$ is perpendicular to the vector $v$ and hence that all points $\alpha_{D_i}(v)$ and $\beta_{D_i}(v)$ lie on the line $L(u, v)$.

*Proof:* In [6, §2] this theorem is proved for the special case that there is only one $D_i$, whose boundary is the unit circle, and with $a = 1/8$, $b = 7/8$, and $c = 2$. But in the proof of [6, Theorem 2.2] the values of $a$, $b$, and $c$ were chosen arbitrarily and the proof works for any $a, b, c$ with $0 < a < b < 1 < c$. We will briefly sketch the proof in [6] and indicate how to modify it to fit the present theorem. Details can be found in Bouhjar [2].

Let $D$ stand for the unit disk centered at $\mathbf{0}$ and let $G$ be a dense $G_\delta$-set in $\mathcal{K}(\partial D)$. Select a sequence $O_0 \supset O_1 \supset \cdots$ of dense open subsets of $\mathcal{K}(\partial D)$ such that $G = \bigcap_{j=0}^\infty O_j$. Let $\mathfrak{C}$ stand for the collection of all semicircles that have $\mathbf{0}$ and a point $u$ with $\|u\| \geq c$ as endpoints. Construct inductively a sequence $B_0, B_1, \ldots$ of compact subsets of the annulus $A = \{u \in \mathbb{R}^2 \colon a \leq \|u\| \leq b\}$ such that every element of $\mathfrak{C}$ intersects every $B_i$ and such that for each integer $j \geq 0$ both $\alpha_D(B_{2j})$ and $\beta_D(B_{2j+1})$ are finite elements of $O_j$. Every $B_i$ consists in fact of a finite union of arcs that are contained in fibres of $\alpha_D$ or $\beta_D$. The reason that when constructing $B_{i+1}$ from $B_i$ we can retain the property that these sets meet every element of $\mathfrak{C}$ is that the radii of the circular arcs that are the fibres of $\alpha_D$ and $\beta_D$ are bounded away from the radii of the semicircles in $\mathfrak{C}$ which allows us to choose the arcs that make up $B_{i+1}$ so close together that they cannot be separated from each other by circles with diameter at least $c$. In Theorem 20 that property is guaranteed by the assumption $M < c$.

The construction in [6] is such that every $B_{i+1}$ can be chosen arbitrarily close to $B_i$ so we can arrange that the $B_i$'s form a Cauchy sequence in $\mathcal{K}(A)$ with limit $K$. This means also that $\alpha_D(B_{i+1})$ and $\beta_D(B_{i+1})$ will be close to $\alpha_D(B_i)$ respectively $\beta_D(B_{i+1})$ and hence we can make sure that $\alpha_D(K)$ and $\beta_D(K)$ are both elements of $G$ by the same method that is used to prove the Baire Category Theorem.

To deal with the infinite sequence $D_1, D_2, \ldots$ instead of a single $D$ let $k(0), k(1), \ldots$ enumerate $\mathbb{N}$ in such a way that $\{j \colon i = k(j)\}$ is infinite for every $i \in \mathbb{N}$. Select for each $i \in \mathbb{N}$ a sequence $O_0^i \supset O_1^i \supset \cdots$ of dense open subsets of $\mathcal{K}(\partial D_i)$ such that $G_i = \bigcap_{j=0}^\infty O_j^i$. We now arrange that for each $j \geq 0$, both $\alpha_{D_{k(j)}}(B_{2j})$ and $\beta_{D_{k(j)}}(B_{2j+1})$ are finite elements of $O_j^{k(j)}$. ∎

Theorem 20 has the following consequence.

PROPOSITION 21: *If $n \geq 2$ is even then there exists a compact partial $n$-point set $Z$ with $\mathcal{H}^1(Z) = 0$ that is not contained in any $n$-point set.*

This result shows that in Theorem 19 we cannot replace the condition $\mathcal{H}^1(Z \times Z) = 0$ by $\mathcal{H}^1(Z) = 0$. In addition, since $\mathcal{H}^1(C) = 0$ implies $\operatorname{ind} C = 0$ we also see that in Theorem 17 it is essential that $Y$ is a partial $(n-1)$-point set and not just a partial $n$-point set.

*Proof:* Let $n = 2k$ and let $C_i = \{u \in \mathbb{R}^2 \colon \|u\| = i\}$ for $i \in \{1, \ldots, k\}$. Since $\mathcal{H}^1$ is an upper semicontinuous function we have that $G_i = \{K \in \mathcal{K}(C_i) \colon \mathcal{H}^1(K) = 0\}$ is a $G_\delta$-set in $\mathcal{K}(G_i)$. Recall that in the Hausdorff metric every compactum can be approximated by its finite subsets so every $G_i$ is dense. According to Theorem 20 we can find for $1 \leq i \leq k$ compact sets $A_i$ and $B_i$ in $G_i$ such that for every point $u$ with $\|u\| \geq k+1$ there is a line $\ell$ through $u$ that meets every $A_i \cup B_i$ in two points. Put $Z = \bigcup_{i=1}^{k}(A_i \cup B_i)$ and note that $\mathcal{H}^1(Z) = 0$. If $Z$ is contained in an $n$-point set $X$ then we consider a line $\ell$ whose distance towards the origin exceeds $k+1$. If $u \in \ell \cap X$ then there is a line through $u$ that contains $n$ points from $Z$ and hence that line contains at least $n+1$ points from $X$, showing that $Z$ is not extendible to an $n$-point set. ∎

It takes a somewhat greater effort to obtain the same result for odd $n$:

PROPOSITION 22: *If $n \geq 3$ is odd then there exists a compact partial $n$-point set $Z$ with $\mathcal{H}^1(Z) = 0$ that is not contained in some $n$-point set.*

*Proof:* Let $n = 2k+1$ and let $C_1, \ldots, C_k$ be circles that are tangent to the line $y = 1$ at the point $q = (0,1)$ with radii that range from 1 to 1.5 as in Figure 15.

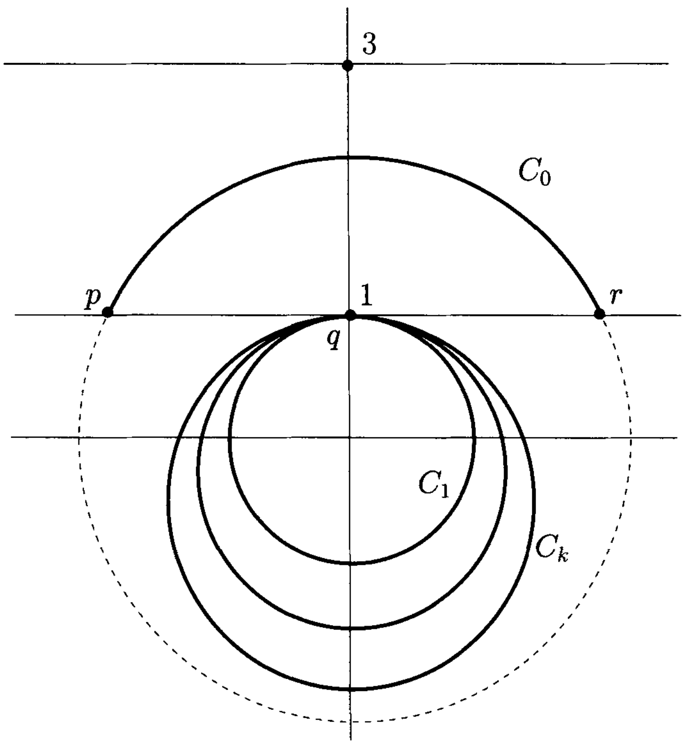


Figure 15

Put $C_{k+1} = \{u \in \mathbb{R}^2\colon \|u\| = \sqrt{5}\}$ and $C_0 = C_{k+1} \cap (\mathbb{R} \times [1,\infty))$. Note that $C_0$ has endpoints $p = (-2,1)$ and $r = (2,1)$. Let $\alpha_i$ and $\beta_i$ be the obvious maps associated with $C_i$ for $1 \le i \le k+1$. Choose with Theorem 20 for every $m \in \mathbb{N}$ a compactum

$$K_m \subset \{u \in \mathbb{R}^2\colon (1+m^{-2})^{-1/2} < \|u\| < 1\}$$

such that $\mathcal{H}^1(\alpha_i(K_m) \cup \beta_i(K_m)) = 0$ for $1 \le i \le k+1$ and such that every semicircle with $\mathbf{0}$ and a point on the line $y = 3$ as endpoints meets $K_m$. Define the compacta $K'_m$ by $K'_1 = K_1$ and, for $m \ge 2$,

$$K'_m = K_m \cap \{(x,y) \in \mathbb{R}^2\colon 3y \ge m|x|\}.$$

The shaded region in Figure 16 represents $K'_m$.

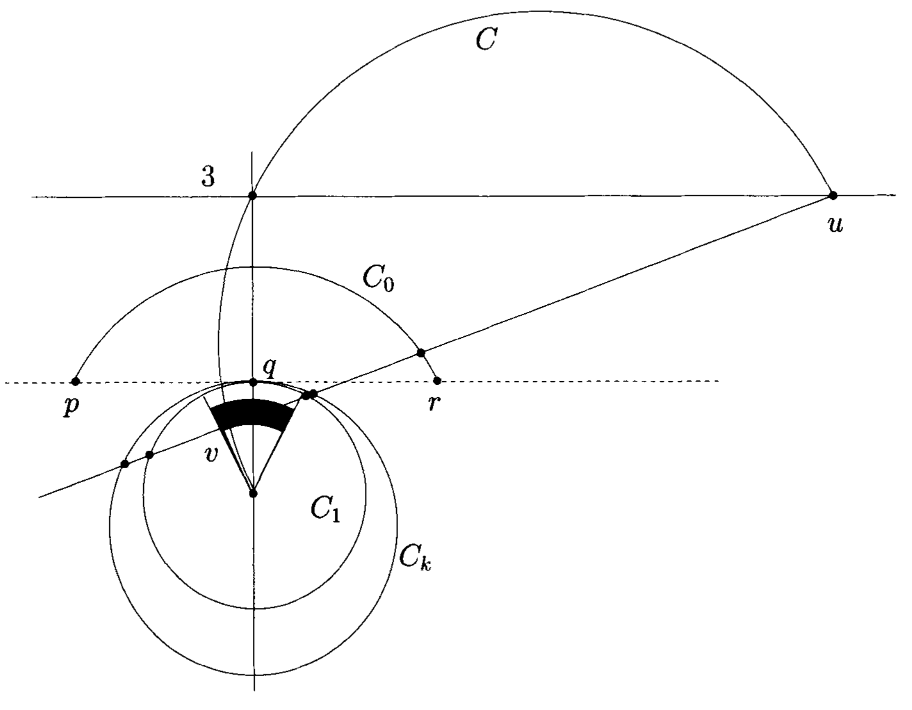


Figure 16

We define for $m \in \mathbb{N}$ the compactum

$$Z_m = (C_0 \cap (\alpha_{k+1}(K'_m) \cup \beta_{k+1}(K'_m))) \cup \bigcup_{i=1}^{k} (\alpha_i(K'_m) \cup \beta_i(K'_m)).$$

Put $Z = \{p,q,r\} \cup \bigcup_{m=1}^{\infty} Z_m$. Note that $\mathcal{H}^1(Z) = 0$ by the $\sigma$-additivity of $\mathcal{H}^1$. In order to show that $Z$ is compact select a convergent sequence $v_2, v_3, \ldots$ such that $v_m \in Z_m$. Then $v = \lim_{m\to\infty} v_m \in \bigcup_{i=1}^{k+1} C_i$. Select for every $m \geq 2$ a $u_m = (x_m, y_m) \in K'_m$ such that some $\alpha_i(u_m)$ or $\beta_i(u_m)$ equals $v_m$. Consequently, $u_m \cdot (v_m - u_m) = 0$. Since $u_m \in K'_m$ we have $(1+m^{-2})^{-1/2} < \|u_m\| < 1$ and hence $\lim_{m\to\infty} \|u_m\| = 1$. We also have

$$\lim_{m\to\infty} |x_m| \leq \lim_{m\to\infty} 3y_m/m \leq \lim_{m\to\infty} 3\|u_m\|/m = 0$$

and hence

$$\lim_{m\to\infty} y_m = \lim_{m\to\infty} \sqrt{\|u_m\|^2 - |x_m|^2} = 1.$$

So we may conclude that $\lim_{m\to\infty} u_m = q$ and hence taking the limit in $u_m \cdot (v_m - u_m) = 0$ produces $q \cdot (v - q) = 0$. This means that $v$ lies on the line $y = 1$ and that $v$ equals $p$, $q$, or $r$. We may conclude that $Z$ is compact.

Assume now that $X$ is an $n$-point set that contains $Z$. Select a $u$ in $X$ that lies on the line $y = 3$. Let $u = (x, 3)$ and consider first the case $|x| \leq 2$. Then both semicircles that have $\mathbf{0}$ and $u$ as endpoints intersect $K_1 = K'_1$ in some point. Call these two points $v_1$ and $v_2$ and note that at least one of the two lines $L(u, v_1)$ and $L(u, v_2)$ does not pass through $q$. Say that $q \notin L(u, v_1)$ and hence that line intersects $\bigcup_{i=1}^{k} C_i$ in $2k$ distinct points that are all contained in $Z_1$. Since $\|v\| < 1$ and $|x| \leq 2$ it is clear that $L(u, v_1)$ also meets $C_0$ in a point of $Z_1$. So $L(u, v_1)$ contains $n+1$ points of $X$, a contradiction.

Consider now the case $|x| \geq 2$. Then $2m-2 \leq |x| \leq 2m$ for some integer $m \geq 2$. Let $C$ be the semicircle with $\mathbf{0}$ and $u$ as endpoints that passes through $(0,3)$; see Figure 16. Then there is a point $v = (x', y')$ in $C \cap K_m$. Observe that $x'$ and $x$ have opposite signs and that the angle between the vectors $u$ and $v$ is acute. So we have $u \cdot v = xx' + 3y' > 0$ and hence $3y' > -xx' = |xx'| \geq (2m-2)|x'| \geq m|x'|$ which means that $v \in K'_m$. Let $b$ be the point on the $y$-axis that lies on the line $L(u, v)$. By similarity of triangles we find $\|v\|/b = |x|/\sqrt{x^2 + (3-b)^2}$. If $b \leq 1$ then

$$\|v\| = \frac{b|x|}{\sqrt{x^2 + (3-b)^2}} \leq \frac{|x|}{\sqrt{x^2+4}} = \frac{1}{\sqrt{1+4x^{-2}}} \leq \frac{1}{\sqrt{1+m^{-2}}}$$

which contradicts the properties of $K_m$. So we have that $b > 1$ which means that the line $L(u, v)$ must intersect $C_0$ in either $\alpha_{k+1}(v)$ or $\beta_{k+1}(v)$. It also means that $L(u, v)$ does not pass through $q$ and hence $L(u, v)$ contains $2k$ points of $Z_m$. Together with $u$ we have a total of $n+1$ points in $X \cap L(u, v)$, contradicting the $n$-point property of $X$. So we may conclude that $Z$ is not extendible to an

$n$-point set. (Note that in Figure 16 we chose for clarity over accuracy, resulting in the false suggestion that $b < 1$.) ∎

If $X$ is a space and $\mathcal{P}$ is a property that can be applied to points of $X$ then we say that **generic** elements of the space have the property $\mathcal{P}$ if there is a dense $G_\delta$-set $G$ of $X$ such that each element of $G$ has the property $\mathcal{P}$. Since $\mathcal{H}^1$ is an upper semicontinuous function we have that $\{K \in \mathcal{K}(X): \mathcal{H}^1(K \times K) = 0\}$ is a $G_\delta$-set in $\mathcal{K}(X)$. Recall that in the Hausdorff metric every compactum can be approximated by its finite subsets. So Theorem 19 implies that under MA generic compact partial $n$-point subsets of $X$ are extendible to $n$-point sets. In Dijkstra [5] it is proved that this result can be obtained in ZFC when $n = 2$. We observe that for the proof in that paper to work it is irrelevant whether one considers two-point sets or $n$-point sets:

PROPOSITION 23: *If $n \geq 2$ and $X \subset \mathbb{R}^2$ then generic compact partial $n$-point subsets of $X$ are extendible to $n$-point sets.*

We conclude this paper by formulating some questions. As mentioned in §2, Theorem 6, Bouhjar, Dijkstra, and Mauldin [3] showed that no $n$-point set is an $F_\sigma$-set in the plane which leads us to:

QUESTION 1: *Can an $n$-point set be a $G_\delta$-set in the plane?*

Dijkstra and van Mill [8] observed that the following result can be obtained as a corollary to the Kuratowski–Ulam Theorem [13].

PROPOSITION 24: *Any partial $n$-point set that is a $G_\delta$-set in $\mathbb{R}^2$ is nowhere dense in $\mathbb{R}^2$.*

It was shown in [4] that no three-point set contains nontrivial continua and that for each $n \geq 4$ there are $n$-point sets that contain arcs. Kulesza [12] proved that every two-point set is zero-dimensional by establishing that all one-dimensional partial two-point sets contain arcs. Note that the structure theorems do not decide the problem whether every three-point set is zero-dimensional. Recently, however, Fearnley, Fearnley and Lamoreaux [11] proved that all three-point sets must indeed be zero-dimensional. This suggest the following

QUESTION 2: *Does every one-dimensional partial $n$-point set contain arcs?*

We call a space $X$ **totally disconnected** if for every pair $x$ and $y$ of distinct points in $X$ there is a clopen set $C$ with $x \in C$ and $y \notin C$. The following observation shows that no one-dimensional partial $n$-point set is totally disconnected.

PROPOSITION 25: *If $X$ is rim-finite and totally disconnected then* $\operatorname{ind} X \leq 0$.

*Proof:* Let $B$ be an arbitrary open set in $X$ with finite boundary $\partial B$ and let $x \in B$. For each $y \in \partial B$ select a clopen set $C_y$ such that $x \in C_y$ and $y \notin C$. It is obvious that $U = B \cap \bigcap_{y \in \partial B} C_y = \overline{B} \cap \bigcap_{y \in \partial B} C_y$ is clopen and that $x \in U \subset B$. ∎

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